CAMBRIDGE ASSESSMENT

STEP Solutions
2011

Mathematics
STEP 9465/9470/9475

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TEST

STEP Mathematics III 2011: Solutions

Section A: Pure Mathematics

1. (i) The differential equation can be solved either by separating variables or using an integrating factor. In either case, $\int \left(\frac{x+2}{x+1}\right) dx$, or the negative of it is required, and this can be found either by re-writing $\left(\frac{x+2}{x+1}\right)$ as $1 + \frac{1}{x+1}$ or using the substitution, $v = x + 1$.
Thus the solution is $u = k(x+1)e^{x}$.

(ii) The substitution $y = ze^{-x}$ yields $\frac{dy}{dx} = z'e^{-x} - ze^{-x}$, and $\frac{d^2y}{dx^2} = z''e^{-x} - 2z'e^{-x} + ze^{-x}$.

Substituting these expressions in the differential equation and simplifying gives

$\big((x+1)z'' - (x+2)z'\big)e^{-x} = 0$ which is effectively the first order differential equation from part (i) with $u = z'$.

So $z' = k(x+1)e^{x}$, which is an exact differential (or integration by parts could be used), $z = kxe^{x} + c$ and so $y = Ax + Be^{-x}$ as required.

(iii) Part (ii)'s substitution gives $z'' - \frac{(x+2)}{(x+1)}z' = (x+1)e^{x}$ which using the integrating factor from part (i) gives $\frac{e^{-x}}{x+1}z' = \int 1dx = x + c$, and thus

$y = (x^2+1) + Ax + Be^{-x}$. Alternatively, the solution to part (ii) is the complementary function and a quadratic particular integral should be conjectured, which in view of the cf need only be $y = Cx^2 + D$, yielding the same result.

2. As $f\left(\frac{p}{q}\right) = 0$, $q^{n-1}f\left(\frac{p}{q}\right) = 0$, which, when evaluated, gives every term but one to be an integer, and so, that term, $\frac{p^n}{q}$, must be an integer, and as *p* and *q* are integers with no common factor greater than 1, this can only happen if $q = 1$, giving the required deduction.

(i) To show that the nth root of 2 is irrational, consider $f(x) = x^n - 2$, and evaluate $f(1)$ and $f(2)$, then apply the stem of the question.

(ii) Considering the turning points of $f(x) = x^3 - x + 1$, there can only be one real root. Evaluating $f(-2)$ and $f(-1)$ and applying the stem gives the result.

(iii) Considering the graphs of $y = x^n$ and $y = 5x - 7$, for $n \geq 3$, that these cannot intersect for $x \geq 0$ can be observed by noting their signs for $0 \leq x < 1 \cdot 4$, and their gradients for $x \geq 1 \cdot 4$. For $x < 0$, and *n* even, it is sufficient to consider signs, whereas for n odd, it is enough to evaluate $f(x) = x^n - 5x + 7$ for $x = -2$, and $x = -1\ or -3$, depending on the case, and then applying the stem. The case $n = 2$, can be demonstrated by completing the square and showing that there are no real roots.

Part (i) could be demonstrated by a minor variant to the usual proof for the irrationality of the square root of 2. Parts (ii) and (iii) could be shown by applying the stem and then considering the left hand side of each equation for the cases *n* even and *n* odd.

3. Considering the quadratic equation $pt^2 - qt + p^2 = 0$, the condition $q^2 \neq 4p^3$ shows, by considering the discriminant, that the roots are unequal. Supposing that $x^3 - 3px + q$ can be written as $a(x-\alpha)^3 + b(x-\beta)^3$, and equating coefficients generates the four equations

$a + b = 1$

$-3\alpha a - 3\beta b = 0$

$3\alpha^2 a + 3\beta^2 b = -3p$

$-\alpha^3 a - \beta^3 b = q$

The first pair may be solved simultaneously to give $a = \frac{-\beta}{\alpha-\beta}$ and $b = \frac{\alpha}{\alpha-\beta}$.

Substitution yields $p = \alpha\beta$ and $q = \alpha\beta(\alpha+\beta)$, or alternatively,

$\alpha\beta = p$ and $\alpha + \beta = \frac{q}{p}$ and so α and β satisfy $t^2 - \frac{q}{p}t + p = 0$ i.e. $pt^2 - qt + p^2 = 0$.

For $p = 8,\ q = 48$, $q^2 - 4p^3 = 2^8 \neq 0$.

Hence α and β are the roots of $8t^2 - 48t + 64 = 0$, i.e. $t^2 - 6t + 8 = 0$ and wlog $\alpha = 2$, $\beta = 4,\ a = 2,\ b = -1$.

So $x^3 - 24x + 48 = 0$ can be re-arranged as $\left(\frac{x-4}{x-2}\right)^3 = 2$

As $\omega^3 = 1$, $\frac{x-4}{x-2} = \sqrt[3]{2}$, $\omega\sqrt[3]{2}$, $\omega^2\sqrt[3]{2}$ and so $x = \frac{2(2-\sqrt[3]{2})}{1-\sqrt[3]{2}}$, $\frac{2(2-\omega\sqrt[3]{2})}{1-\omega\sqrt[3]{2}}$, $\frac{2(2-\omega^2\sqrt[3]{2})}{1-\omega^2\sqrt[3]{2}}$

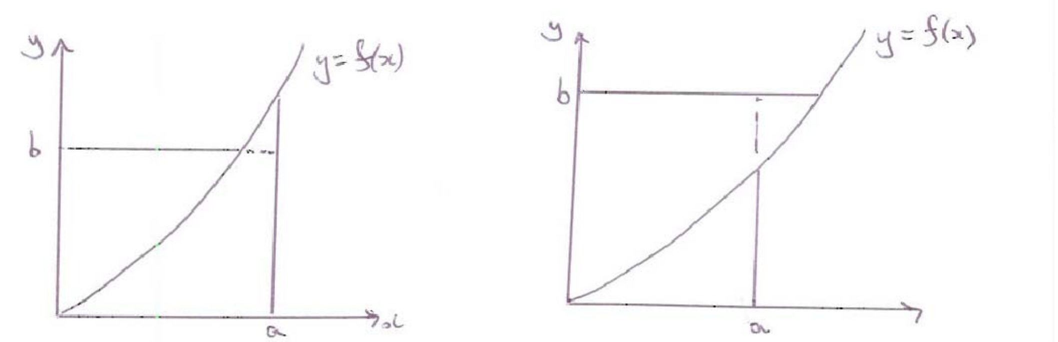
If $q = 2r^3$ and $p = r^2$, $q^2 = 4p^3$ so the first part cannot be used.

However, $x^3 - 3r^2x + 2r^3 = 0$ can be readily factorised as $(x-r)^2(x+2r) = 0$ and so $x = r$ (repeated) or $-2r$

4. (i) $\int_0^a f(x)dx$ is the area between the curve $y = f(x)$, the *x* axis, and the line $x = a$

$\int_0^b f^{-1}(y)dy$ is the area between the curve $y = f(x)$, the *y* axis, and the line $y = b$.

The sum of these areas is greater than or equal to the area of the rectangle, with equality holding if $b = f(a)$.



(ii) With $(x) = x^{p-1}$, the sum of the two integrals is $\frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}}$

But as $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{q}=\frac{p-1}{p}$, and so the required result follows by applying the result of part (i).
If $b=a^{p-1}$, simple algebra shows $a=b^{q-1}$, so $\frac{1}{p}a^p+\frac{1}{q}b^q=\frac{1}{p}ab+\frac{1}{q}ba=ab$ and equality is verified.

(iii) $f(x)=\sin x$ satisfies the conditions of part (i)
So $\int_0^a f(x)\,dx=1-\cos a$, and, by parts, $\int_0^b f^{-1}(y)dy=b\sin^{-1}b+\sqrt{1-b^2}-1$ which together give the required result.

Choosing $a=0$, and $b=t^{-1}$, part (i) gives $0\le t^{-1}\sin^{-1}(t^{-1})+\sqrt{1-t^{-2}}-1$ which can be re-arranged to give the required result.

5.
$$r^2d\theta=(x^2+y^2)\frac{d}{dt}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)dt=(x^2+y^2)\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)}{x^2}dt\;=\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$
and hence integrating gives the result.

A is $(x-a\cos t,y-a\sin t)$ and B is $(x+b\cos t,y+b\sin t)$

$[A]=\frac{1}{2}\int_0^{2\pi}(x-a\cos t)\left(\frac{dy}{dt}-a\cos t\right)-(y-a\sin t)\left(\frac{dx}{dt}+a\sin t\right)dt$ using (*) which leads directly to $[A]=[P]-af+\pi a^2$.
Replacing $-a$ by b gives $[B]=[P]+bf+\pi b^2$
As $[A]=[B]$, these expressions can be equated to give $f=\pi(a-b)$.
The area between curves C and D is $[A]-[P]=-af+\pi a^2$ which by substitution gives πab as required.

6. Using the substitution $t=\tanh\left(\frac{u}{2}\right)$, then it can be shown that $T=U$, by making use of $2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2}\right)=\sinh u$ to obtain the integrand, and $\tanh^{-1}t=\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ to obtain the limits.
If instead, integration by parts is used differentiating $\tanh^{-1}t$ and integrating $\frac{1}{t}$,and employing $\tanh^{-1}t=\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ to demonstrate that $[\tanh^{-1}t\ln t]_{\frac{1}{3}}^{\frac{1}{2}}=0$, $T=V$.
The substitution $t=e^{-2x}$ can be used to demonstrate that $T=X$.

(Alternatively, starting from U, the substitution $u=2\tanh^{-1}t$ obtains $U=T$, the substitution $u=-\ln v$ obtains $U=V$, and the substitution $u=2x$ followed by integration by parts yields $U=X$; starting from V, by parts it can be shown that $V=T$, using the substitution $v=e^{-u}$ that $V=U$, and the substitution $v=\tanh x$ that $V=X$; or starting from X, the substitution $x=-\frac{1}{2}\ln t$ gives $X=T$, integration by parts gives $X=U$, and the substitution $x=\tanh^{-1}v$ gives $X=V$.)

7. (i) The induction requires $T_{k+2}=A_{k+2}+B_{k+2}\sqrt{a(a+1)}$ and ${A_{k+2}}^2-a(a+1){B_{k+2}}^2=1$.
$$T_{k+2}=\left(A_k+B_k\sqrt{a(a+1)}\right)\left(\sqrt{a+1}+\sqrt{a}\right)^2=\left(A_k+B_k\sqrt{a(a+1)}\right)T_2$$

$T_2 = \left(2a+1+2\sqrt{a(a+1)}\right)$ and so $A_2 = 2a+1$ and $B_2 = 2$, and
${A_2}^2 - a(a+1){B_2}^2 = (2a+1)^2 - a(a+1)2^2 = 1$ the result is true for $n = 2$.
Evaluating T_{k+2} using $\left(A_k + B_k\sqrt{a(a+1)}\right)T_2$ then $A_{k+2} = (2a+1)A_k + 2a(a+1)B_k$
and $B_{k+2} = 2A_k + (2a+1)B_k$, and so substituting and simplifying,
${A_{k+2}}^2 - a(a+1){B_{k+2}}^2 = {A_k}^2 - a(a+1){B_k}^2 = 1$ by the induction.

(ii) $T_n = (\sqrt{a+1}+\sqrt{a})T_m = (\sqrt{a+1}+\sqrt{a})\left(A_m + B_m\sqrt{a(a+1)}\right)$

$= (A_m + aB_m)\sqrt{a+1} + (A_m + (a+1)B_m)\sqrt{a}$ which is of required form because

$C_n = A_m + aB_m$ and $D_n = A_m + (a+1)B_m$ are integers and

$(a+1){C_n}^2 - a{D_n}^2 = (a+1)(A_m + aB_m)^2 - a(A_m + (a+1)B_m)^2$

$= {A_m}^2 - a(a+1){B_m}^2 = 1$ as required.

Trivially the case $n = 1$ is true.

(iii) In the case that *n* is even,
$T_n = A_n + B_n\sqrt{a(a+1)} = \sqrt{{A_n}^2} + \sqrt{a(a+1){B_n}^2} = \sqrt{a(a+1){B_n}^2+1} + \sqrt{a(a+1){B_n}^2}$
as required,
and in the case that n is odd, $T_n = C_n\sqrt{a+1} + D_n\sqrt{a} = \sqrt{(a+1){C_n}^2} + \sqrt{a{D_n}^2} =$
$\sqrt{a{D_n}^2+1} + \sqrt{a{D_n}^2}$ as required.

8. $w = u + iv = \frac{1+i(x+iy)}{i+(x+iy)} = \frac{2x}{x^2+(1+y)^2} + i\frac{x^2-(1-y^2)}{x^2+(1+y)^2}$ using the complex conjugate, so
$u = \frac{2x}{x^2+(1+y)^2}$ and $v = \frac{x^2-(1-y^2)}{x^2+(1+y)^2}$

(i) If $x = \tan\frac{\theta}{2}$, $y = 0$, then $u = \sin\theta$, and $v = -\cos\theta$, using the general result and so $u^2 + v^2 = 1$ but the point $\theta = \pi$ i.e. (0,1) is not included.

(ii) If $-1 < x < 1$, and $y = 0$, then it is the same locus as (i) except $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and so it is the semi-circle that is the part of $u^2 + v^2 = 1$ below the *u* axis.

(iii) $x = 0$, then $u = 0$ and $v = \frac{y-1}{y+1}$, and as $-1 < y < 1$ which is that part of the *v* axis below the *u* axis, i.e. $-\infty < v < 0$.

(iv) Let $x = 2\tan\frac{\theta}{2}$ and $y = 1$, so as $-\infty < x < \infty$, $-\pi < \theta < \pi$, then
$u = \frac{1}{2}\sin\theta$ and $v = \frac{1}{2}(1-\cos\theta)$, so the locus is the circle $u^2 + \left(v - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$
excluding the point $\theta = \pi$, which is $(0,1)$.

Section B: Mechanics

9. For the initial equilibrium position, suppose $\theta = \alpha$, considering potential energy, with potential energy zero level at O, $U = 4mga\cos\theta + 3mga\sin\theta + c$, for equilibrium, $\frac{dU}{d\theta} = 0$, giving $\tan\alpha = \frac{3}{4}$.
Then conserving energy,
$4mga\cos\theta + 3mga\sin\theta + \frac{1}{2}\,7m\left(a\dot{\theta}\right)^2 = 4mga\cos\alpha + 3mga\sin\alpha$ which having substituted for α gives $7a\left(\dot{\theta}\right)^2 + 8g\cos\theta + 6g\sin\theta = 10g$

(i) Resolving radially in general for Q, if R is the contact force,
$4mg\cos\theta - R = 4ma\dot{\theta}^2$ **,** so when $\theta = \beta$, $R = 0$, and thus $4mg\cos\beta = 4ma\dot{\theta}^2$ and so substituting for $\dot{\theta}^2$ and θ in the energy result gives $15\cos\beta + 6\sin\beta = 10$.

(ii) Resolving tangentially for Q, $4mg\sin\theta - T = 4ma\ddot{\theta}$ and for P,
$T - 3mg\cos\theta = 3ma\ddot{\theta}$ so eliminating $\ddot{\theta}$ between them and re-arranging,
$T = \frac{12}{7}mg(\sin\theta + \cos\theta)$ as required.

10. Suppose Q is displaced x and P is displaced y, and let $\lambda = \frac{1}{2}ma\omega^2$,
then $m\ddot{x} = \frac{\lambda(y-x)}{a}$ and $m\ddot{y} = \frac{-\lambda(y-x)}{a}$.
Adding and integrating leads to $x + y = ut$.
Subtracting gives $\ddot{y} - \ddot{x} = -\omega^2 (y - x)$ and so $y - x = \frac{u}{\omega}\sin\omega t$ from solving the differential equation and employing the initial conditions that when = 0, $x = y = 0, \dot{x} = 0$, and $\dot{y} = u$.
Thus, $x = \frac{1}{2}\left(ut - \frac{u}{\omega}\sin\omega t\right)$ and $y = \frac{1}{2}\left(ut + \frac{u}{\omega}\sin\omega t\right)$. When the string next returns to length a , $y - x = \frac{u}{\omega}\sin\omega t = 0$, $\omega t = \pi$ and so $x = y = \frac{1}{2}\,\frac{u}{\omega}\pi$ as required.
So at this time, $\dot{x} = u$, and $\dot{y} = 0$.
The total time between the impulse and the subsequent collision is $\frac{\pi}{\omega} + \frac{a}{u}$.

11. On the one hand the distance between the point on the disc vertically below $(a, 0,0)$ and P is $b\sin\phi$ as the string length b makes an angle ϕ with the vertical. On the other, it is $2a\sin\frac{1}{2}\theta$, the third side of an isosceles triangle with two radii a at an angle θ , and hence the required result.
The horizontal component of the tension in each string is $T\sin\phi$ and it acts at a perpendicular distance $a\cos\frac{1}{2}\theta$ from the axis of symmetry. Thus the couple is $nT\sin\phi\; a\cos\frac{1}{2}\theta$. Resolving vertically, $nT\cos\phi = mg$. Substituting for T in the expression for the couple and then using $b\sin\phi = 2a\sin\frac{1}{2}\theta$ to eliminate ϕ , gives the required result.
The initial potential energy relative to the position where the strings are vertical is $mgb(1 - \cos\phi)$. This is converted into kinetic energy $\frac{1}{2}\,\frac{1}{2}\,ma^2\omega^2$. Equating these expressions and once again using $b\sin\phi = 2a\sin\frac{1}{2}\theta$ to eliminate ϕ , gives the required result.

Section C: Probability and Statistics

12. As $G_Y(t) = G\big(H(t)\big)$, $G'_Y(t) = G'\big(H(t)\big) \times H'(t)$, and as $H(1) = 1$, $H'(1) = E(X_i)$, $G'(1) = E(N)$, and $G'_Y(1) = E(Y)$, the first result follows.

Similarly, $G''_Y(t) = G''\big(H(t)\big) \times \big(H'(t)\big)^2 + G'\big(H(t)\big) \times H''(t)$, and

$Var(Y) = G''_Y(1) + G'_Y(1) - \big(G'_Y(1)\big)^2$

$= G''\big(H(1)\big) \times \big(H'(1)\big)^2 + G'\big(H(1)\big) \times H''(1) + E(Y) - \big(E(Y)\big)^2$

$= \Big(Var(N) + \big(E(N)\big)^2 - E(N)\Big) \times \big(E(X_i)\big)^2 + E(N) \times \Big(Var(X_i) + \big(E(X_i)\big)^2 - E(X_i)\Big) + E(N)\,E(X_i) - \big(E(N)\,E(X_i)\big)^2$

$= Var(N) \times \big(E(X_i)\big)^2 + E(N) \times Var(X_i)$ as required.

A fair coin tossed until a head appears is distributed $Geo\left(\frac{1}{2}\right)$ so $G(t) = \frac{t}{2-t}$. The PGF for the number of heads when a fair coin is tossed once is $\frac{1}{2} + \frac{1}{2}t$. Thus $G_Y(t) = \frac{1+t}{3-t}$.

Using the results $E(Y) = 2 \times \frac{1}{2} = 1$, and $Var(Y) = \frac{1-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} \times \left(\frac{1}{2}\right)^2 + 2 \times \frac{1}{4} = 1$.

$P(Y = r)$, being the coefficient of t^r in $G_Y(t)$, is $\frac{4}{3^{r+1}}$ for $r \geq 1$, and $\frac{1}{3}$ for $r = 0$.

13. (i) $P(X = r) = \binom{k}{r}\left(\frac{b}{n}\right)^r\left(\frac{n-b}{n}\right)^{k-r}$, $P(X = r+1) = \binom{k}{r+1}\left(\frac{b}{n}\right)^{r+1}\left(\frac{n-b}{n}\right)^{k-r-1}$

and so $\frac{P(X=r+1)}{P(X=r)} = \frac{k-r}{r+1}\frac{b}{n-b}$. The most probable value of *X* is the minimum value of *r* such that $\frac{k-r}{r+1}\frac{b}{n-b} < 1$, because this expression decreases as r increases. All the factors are positive so it is simple to rearrange the algebra to obtain $r > \frac{(k+1)}{n}b - 1$ so $r = \left\lfloor \frac{(k+1)}{n}b \right\rfloor$.

The answer is not unique when there is a value of r such that $\frac{k-r}{r+1}\frac{b}{n-b} = 1$, in which case, $= \frac{(k+1)}{n}b$, which will only happen if n divides $(k+1)b$.

(ii) Using the same strategy as for part (i), $P(X = r) = \frac{\binom{b}{r}\binom{n-b}{k-r}}{\binom{n}{k}}$,

$P(X = r+1) = \frac{\binom{b}{r+1}\binom{n-b}{k-r-1}}{\binom{n}{k}}$, and so $\frac{P(X=r+1)}{P(X=r)} = \frac{k-r}{r+1}\ \frac{b-r}{n-b-(k-r)+1}$.

Again, the most probable value of *X* is the minimum value of *r* such that $\frac{k-r}{r+1}\frac{b-r}{n-b-(k-r)+1} < 1$, giving $r = \left\lfloor \frac{(k+1)(b+1)}{(n+2)} \right\rfloor$, and this is not unique if $(n+2)$ divides $(k+1)(b+1)$.